

**3462.** [2009 : 327, 329] *Proposed by Sotiris Louridas, Aegaleo, Greece.*

Let  $x$ ,  $y$ , and  $z$  be positive real numbers such that

$$(x^3 + z^3 - y^3)(y^3 + x^3 - z^3)(z^3 + y^3 - x^3) > 0.$$

Prove that

$$\begin{aligned} (x^3 + y^3 + z^3 + 3xyz) \prod_{\text{cyclic}} (x^3 + y^3 - z^3 + xyz) \\ \leq 3 \prod_{\text{cyclic}} \sqrt[3]{x^4(x^2 + yz)^4}. \end{aligned}$$

*Composite of similar solutions by Arkady Alt, San Jose, CA, USA, and Thanos Magkos, 3<sup>rd</sup> High School of Kozani, Kozani, Greece, modified by the editor.*

First note that the hypotheses imply that each of the terms  $x^3 + y^3 - z^3$ ,  $y^3 + z^3 - x^3$ , and  $z^3 + x^3 - y^3$  is positive, since if two of them are negative, say  $x^3 + y^3 - z^3 < 0$  and  $y^3 + z^3 - x^3 < 0$ , then we would have  $2y^3 < 0$ , or  $y < 0$ , a contradiction. Hence, if we set  $a = x^3 + 3xyz$ ,  $b = y^3 + 3xyz$ , and  $c = z^3 + 3xyz$ , then  $a + b - c = x^3 + y^3 - z^3 + 3xyz > 0$ , which implies that  $a + b > c$ . Similarly,  $b + c > a$  and  $c + a > b$ . Therefore, since  $a$ ,  $b$ , and  $c$  are positive, they are the side lengths of a triangle  $ABC$ . In this context, the inequality to be proved is now rewritten as

$$(a + b + c)(a + b - c)(b + c - a)(c + a - b) \leq 3\sqrt[3]{a^4b^4c^4}. \quad (1)$$

Let  $s$ ,  $R$ , and  $F$  denote the semiperimeter, the circumradius, and the area of triangle  $ABC$ . The following formulas are well known:

$$\begin{aligned} F &= \sqrt{s(s-a)(s-b)(s-c)}; \\ \frac{a}{\sin A} &= \frac{b}{\sin B} = \frac{c}{\sin C} = 2R; \\ abc &= 4RF. \end{aligned}$$

Hence, inequality (1) is equivalent to each of the following:

$$\begin{aligned} 16s(s-a)(s-b)(s-c) &\leq 3\sqrt[3]{(4RF)^4}, \\ 16^3 F^6 &\leq 3^3 \cdot 4^4 \cdot R^4 \cdot F^4, \\ abc = 4RF &\leq 3\sqrt[3]{3}R^3, \\ 8(\sin A \sin B \sin C)R^3 &\leq 3\sqrt[3]{3}R^3, \\ \sin A \sin B \sin C &\leq \frac{3\sqrt[3]{3}}{8}, \end{aligned}$$

and it is well known that the last inequality is true [Ed: *c.f. Formula 2.8 on p. 20 of O. Bottema et al., Geometric Inequalities, Wolters-Noordhoff Publ., Groningen, 1969.*]

Thus, inequality (1) is established, and the problem is solved.

*Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer. There was one incorrect solution submitted.*

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