**3462**. [2009 : 327, 329] *Proposed by Sotiris Louridas, Aegaleo, Greece.*

Let $x$, $y$, and $z$ be positive real numbers such that

$$\left(x^3+z^3-y^3\right)\left(y^3+x^3-z^3\right)\left(z^3+y^3-x^3\right) > 0\,.$$

Prove that

$$\left(x^3+y^3+z^3+3xyz\right)\prod_{\text{cyclic}}\left(x^3+y^3-z^3+xyz\right) \le 3\prod_{\text{cyclic}}\sqrt[3]{x^4\left(x^2+yz\right)^4}\,.$$

*Composite of similar solutions by Arkady Alt, San Jose, CA, USA, and Thanos Magkos, 3*[rd] *High School of Kozani, Kozani, Greece, modified by the editor.*

First note that the hypotheses imply that each of the terms $x^3+y^3-z^3$, $y^3+z^3-x^3$, and $z^3+x^3-y^3$ is positive, since if two of them are negative, say $x^3+y^3-z^3<0$ and $y^3+z^3-x^3<0$, then we would have $2y^3<0$, or $y<0$, a contradiction. Hence, if we set $a=x^3+3xyz$, $b=y^3+3xyz$, and $c=z^3+3xyz$, then $a+b-c=x^3+y^3-z^3+3xyz>0$, which implies that $a+b>c$. Similarly, $b+c>a$ and $c+a>b$. Therefore, since $a$, $b$, and $c$ are positive, they are the side lengths of a triangle $ABC$. In this context, the inequality to be proved is now rewritten as

$$(a+b+c)(a+b-c)(b+c-a)(c+a-b) \le 3\sqrt[3]{a^4b^4c^4}\,. \qquad (1)$$

Let $s$, $R$, and $F$ denote the semiperimeter, the circumradius, and the area of triangle $ABC$. The following formulas are well known:

$$\begin{aligned} F &= \sqrt{s(s-a)(s-b)(s-c)}\,; \\ \frac{a}{\sin A} &= \frac{b}{\sin B} = \frac{c}{\sin C} = 2R\,; \\ abc &= 4RF\,. \end{aligned}$$

Hence, inequality (1) is equivalent to each of the following:

$$\begin{aligned} 16s(s-a)(s-b)(s-c) &\le 3\sqrt[3]{(4RF)^4}\,, \\ 16^3F^6 &\le 3^3\cdot 4^4\cdot R^4\cdot F^4\,, \\ abc = 4RF &\le 3\sqrt{3}R^3\,, \\ 8(\sin A\sin B\sin C)R^3 &\le 3\sqrt{3}R^3\,, \\ \sin A\sin B\sin C &\le \frac{3\sqrt{3}}{8}\,, \end{aligned}$$

and it is well known that the last inequality is true [*Ed: c.f. Formula 2.8 on p. 20 of O. Bottema etal., Geometric Inequalities, Wolters-Noordhoff Publ., Groningen, 1969.*]

Thus, inequality (1) is established, and the problem is solved.

*Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer. There was one incorrect solution submitted.*